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Explicit Computation of Padé–Hermite Approximants

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Using a method of Siegel and the q-derivation, we compute explicitly the Padé–Hermite approximants of a system of functions connected with the q-logarithm $L_q(x) = \sum x^n/(q^n - 1)$.

1. INTRODUCTION

The purpose of this paper is to compute explicitly a part of the table of the Padé–Hermite approximants of a system of functions connected with *the q-logarithmic series*:

$$L_o(x) = \sum_{n=0}^{+\infty} \frac{q-1}{q^{n+1}-1} x^{n+1}. \tag{1}$$

By *the Padé–Hermite approximants* of a system of formal series $f_1(x), f_2(x), ..., f_m(x)$ with coefficients in an arbitrary commutative field K, we mean a family of m polynomials $P_1, P_2, ..., P_m$ of respective degrees $\rho_1, \rho_2, ..., \rho_m$ such that:

$$\sum_{i=1}^{m} P_i(x)\, f_i(x) = x^{\rho+m-1} R(x) \tag{2}$$

with $\rho = \sum_{i=1}^{m} \rho_i$ and $R(x) \in K[[x]]$.

Such approximants are very useful to prove linear independence results over $\mathbb{Q}$ in number theory. They were called Padé-approximants of type I by Mahler [11], who succeeded in computing them for a wide class of functions in the case $K = \mathbb{C}$ by means of the residue theorem. In a series of papers dating back to 1964 [10], Jager enlarged the algebraic part of Mahler's work and gave the complete Padé–Hermite table for the two following systems of functions:

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(a) *The binomial function system*: for $i=1, 2, ..., m$, $f_i(x)=(1-x)^{\omega_i}$, with $\omega_i-\omega_j \notin \mathbb{Z}$ if $i \neq j$.

(b) *The exponential function system*: for $i=1, 2, ..., m$, $f_i(x)=\exp(\omega_i x)$, $\omega_i \neq \omega_j$ if $i \neq j$.

For $\rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_m$, Jager also provided the Padé–Hermite approximants of *the logarithmic function system*: $f_i(x)=\operatorname{Log}^{m-i}(1-x)$ for $i=1, 2, ..., m$.

More recently, Borwein in [1] used the same method to compute the ordinary Padé table ($m=2$, $f_1(x)=1$) of complex functions satisfying Poincaré-type equations:

$$f(qx)=(ax+b)\, f(x)+cx+d. \tag{3}$$

As a striking application, Borwein proved the irrationality of $\sum_{n=1}^{+\infty} (1/(q^n+r))$, $q \in \mathbb{Z}$, $|q| \geqslant 2$, $r \in \mathbb{Q}^*$ (see [2] and [3]).

It is not difficult to see that the residue theorem allows, in fact, the explicit computation of the Padé–Hermite table for any system $f_i(x) = f(\omega_i x)$, when $\omega_1 = 0$, $\omega_i/w_j \neq q^p$ $(p \in \mathbb{Z})$ if $i > j > 0$, and $\rho_1 \geqslant \max(\rho_2, ..., \rho_m)$, if f satisfies a Poincaré-type equation like (3). One only has to consider the complex integral:

$$R(x)=\frac{1}{2i\pi}\int_{\mathscr{C}} \frac{f(tx)}{t^{\rho_1}\prod_{k=2}^{m}\prod_{\nu=0}^{\rho_k-1}(t-\omega_k q^\nu)}\, dt, \tag{4}$$

where $\mathscr{C}$ is a positive contour enclosing all the simple poles $\omega_k q^\nu$ of the integrand, as well as zero.

Another method to compute explicitly the Padé–Hermite table of the exponential system was introduced by Siegel [12, Chap. 1]; see also [13, Chap. 2]. Differentiating (1) ρ_1-times, he succeeded in obtaining a recurrence relation over m, thus computing the P_i's and $R(x)$ (as a multiple integral).

The same method was used later by Wallisser in the case of the ordinary Padé table of the q-exponential function [14] (Wallisser replaced the ordinary derivation by the q-derivation), and also by Huttner [9] in the case $f_1(x)=1$; $f_2(x)=\operatorname{Log}(1-x)$: $f_3(x)=\sum_{n=1}^{+\infty} (x^n/n^2)$ (dilogarithmic function).

In this paper, we will use Siegel's method, together with the q-derivation, to compute the diagonal ($\rho_i=\rho_j$ for every i, j) of the Padé–Hermite table of the system $\{1, L_i(\alpha_j x)\colon i=0, 1, ..., k; j=1, 2, ..., N\}$, where:

$$L_k(x)=\sum_{n=0}^{+\infty}\binom{k+n}{k}\frac{q-1}{q^{n+1}-1}\, x^{n+1}. \tag{5}$$

It is readily seen that every series of the form

$$f(x) = \sum_{n=1}^{+\infty} P(n) \frac{x^{n+1}}{q^{n+1}-1}, \tag{6}$$

where P is a polynomial, is a linear combination of the L_k's. This holds, in particular, for the (ordinary) derivatives of the q-logarithmic function, and for the eighth power of the θ_3 Jacobi's function as well, because

$$\left(\sum_{-\infty}^{+\infty} q^{-n^2}\right)^8 = 1 + 16 \sum_{n=1}^{+\infty} \frac{n^3}{1-(-q)^n} \qquad [8, \text{ p. } 315].$$

Our main result is Theorem 1; it will be proved in Section 3. Section 2 is devoted to technical preliminaries. In Section 4, we will study the case $K = \mathbb{C}$ and give the expression of $R(x)$ in (2) as a complex integral similar to (4).

2. THE SERIES $L_k(x)$

(a) Let K be a commutative field, $\text{char}(K) = 0$. Let $q \in K^*$, with $q^n \neq 1$, $\forall n \in \mathbb{N} - \{0\}$.

We denote by δ_q the *q-derivation* in $K[[x]]$, the ring of the formal series with coefficients in K; if $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, we put

$$\delta_q f(x) = \sum_{n=1}^{+\infty} a_n \frac{q^n - 1}{q-1} x^{n-1}. \tag{7}$$

The q-derivation is a classical special case [6] of the U-derivation ([4, 5].

It is easy to verify that:

$$\delta_q f(x) = \frac{f(qx) - f(x)}{x(q-1)}. \tag{8}$$

Leibniz's rule for the q-derivation may be written:

$$\delta_q^n (fg)(x) = \sum_{k=0}^{n} \binom{n}{k}_q \delta_q^{n-k} f(q^k x)\, \delta_q^k g(x). \tag{9}$$

In (9) the *q-binomial coefficients* are defined by

$$\binom{n}{k}_q = \frac{n_q!}{k_q!(n-k)_q!}, \tag{10}$$

with

$$\begin{cases} n_q! = \prod_{m=1}^{n} \frac{q^m - 1}{q - 1} & \text{when} \quad n \geqslant 1. \\ 0_q! = 1. \end{cases}$$

Cauchy's q-binomial theorem [6, 7] asserts that:

$$\prod_{i=1}^{n} (1 - xq^i) = \sum_{p=0}^{n} (-1)^p \binom{n}{p}_q q^{p(p+1)/2} x^p. \tag{12}$$

(b) Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n \in K[[x]]$. We put

$$\int_0^x f(t)\, d_q t = \sum_{n=0}^{+\infty} a_n \frac{q-1}{q^{n+1}-1} x^{n+1}. \tag{13}$$

The properties of the q-integrals are well-known [6, 7] and easy to prove. We will use two of them:

The formula of q-integration by parts:

$$\int_0^x (\delta_q f(t))\, g(t)\, d_q t = [(fg)(t)]_0^x - \int_0^x f(qt)\, \delta_q g(t)\, d_q t. \tag{14}$$

The change of variable $t = au$:

$$\int_0^x f(t)\, d_q t = a \int_0^{a^{-1}x} f(au)\, d_q u. \tag{15}$$

(c) Now let $f \in K[t][[x]]$ be a formal series whose coefficients are polynomials in t. If $f(t, x) = \sum_{n=0}^{+\infty} P_n(t)\, x^n$, we put:

$$\int_0^1 f(t, x)\, d_q t = \sum_{n=0}^{+\infty} \left(\int_0^1 P_n(t)\, d_q t \right) x^n. \tag{16}$$

LEMMA 1 (*q-Taylor's Formula with Integral Remainder*). *Let* $H_0(x, u) = 1$ *and* $H_n(x, u) = \prod_{k=1}^{n} (x - q^k u)$ *for* $n \geqslant 1$. *Then*, $\forall n \in \mathbb{N}$:

$$f(x) = \sum_{k=0}^{n} \frac{\delta_q^k f(0)}{k_q!} x^k + \int_0^x \frac{H_n(x, u)}{n_q!} \delta_q^{n+1} f(u)\, d_q u.$$

Proof. The formula is clearly true for $n = 0$. Suppose it is true for $n - 1$, and put:

$$G_n(x, u) = (x - u)(x - qu) \cdots (x - q^{n-1}u).$$

We have $H_n(x, u) = G_n(x, qu)$, and an easy computation shows that

$$\delta_q G_n(x, u) = -\frac{q^n - 1}{q - 1} H_{n-1}(x, u),$$

Using the formula of q-integration by parts, we obtain for $n \geqslant 1$:

$$\begin{aligned}\int_0^x \frac{H_n(x, u)}{n_q!} \delta_q^{n+1} f(u)\, d_q u &= \int_0^x \frac{G_n(x, qu)}{n_q!} \delta_q^{n+1} f(u)\, d_q u \\ &= \left[\frac{G_n(x, u)}{n_q!} \delta_q^n f(u) \right]_0^x \\ &\quad + \int_0^x \frac{q^n - 1}{q - 1} \frac{H_{n-1}(x, u)}{n_q!} \delta_q^n f(u)\, d_q u \\ &= -\frac{x^n}{n_q!} \delta_q^n f(0) + \int_0^x \frac{H_{n-1}(x, u)}{(n-1)_q!} \delta_q^n f(u)\, d_q u.\end{aligned}$$

Thus Lemma 1 is proved by induction.

LEMMA 2. *Let* $F \in K[x]$, *satisfying*:

(i) $\delta_q^{n+1} F(x) = f(x)$.

(ii) $\delta_q^k F(0) = 0$ *for* $k = 0, 1, ..., n$.

Then $F(x) = (x^{n+1}/n_q!) \int_0^1 (1 - qt)(1 - q^2 t) \cdots (1 - q^n t)\, f(tx)\, d_q t$.

Proof. This is Lemma 1, where f is replaced by F. We also performed the change of variable $u = xt$.

(d) We put, for $k \in \mathbb{N}$:

$$L_k(x) = \int_0^x \frac{d_q t}{(1 - t)^{k+1}} = (q - 1) \sum_{n=0}^{+\infty} \binom{k+n}{k} \frac{x^{n+1}}{q^{n+1} - 1}. \tag{17}$$

LEMMA 3. $\forall \nu \in \mathbb{N}$:

$$L_k(q^\nu x) = L_k(x) + (q - 1) \sum_{\zeta=0}^{\nu-1} \frac{q^\zeta x}{(1 - q^\zeta x)^{k+1}}.$$

Proof. We have:

$$\delta_q L_k(x) = \frac{1}{(1 - x)^{k+1}} = \frac{L_k(qx) - L_k(x)}{x(q - 1)} \qquad \text{by (8).}$$

Thus,

$$L_k(qx) = L_k(x) + x(q-1)\,\frac{1}{(1-x)^{k+1}}. \tag{18}$$

Lemma 3 follows from (18) by an easy induction.

LEMMA 4. $\forall j \in \mathbb{N} - \{0\}$:

$$\delta_q^j L_k(x) = \frac{H_{jk}(x)}{((1-x)(1-qx)\cdots(1-q^{j-1}x))^{k+1}}, \tag{19}$$

with $H_{jk}(x) \in K[x]$, deg $H_{jk} \leqslant k(j-1)$.

Proof. Equation (19) is true for $j=1$: in this case $H_{j1}=1$. Suppose (19) is true for j, and compute:

$$\begin{aligned}\delta_q^{j+1} L_k(x) &= \frac{1}{x(q-1)}\left(\frac{H_{jk}(qx)}{(\prod_{i=1}^{j}(1-q^i x))^{k+1}} - \frac{H_{jk}(x)}{(\prod_{i=0}^{j-1}(1-q^i x))^{k+1}}\right)\\ &= \frac{1}{x(q-1)}\,\frac{H_{jk}(qx)(1-x)^{k+1} - H_{jk}(x)(1-q^j x)^{k+1}}{(\prod_{i=0}^{j}(1-q^i x))^{k+1}}\end{aligned}$$

It is easy to see that $(1/x(q-1))(H_{jk}(qx)(1-x)^{k+1} - H_{jk}(x)\,(1-q^j x)^{k+1})$ is a polynomial $H_{j+1,k}(x)$, with:

$$\deg H_{j+1,k} \leqslant k+1+\deg H_{jk} - 1 \leqslant k+1+k(j-1)-1.$$

Lemma 4 is proved by induction.

LEMMA 5. *Let* $\alpha_1, \alpha_2, ..., \alpha_N \in K^*$, *and let* $k \in \mathbb{N}$. *Let* $G = \{q^n : n \in \mathbb{Z}\}$. *Then* 1 *and the* $L_i(\alpha_m x)$ $(0 \leqslant i \leqslant k;\ 1 \leqslant m \leqslant N)$ *are linearly dependent over the field* $K(x)$ *of the rational fractions with coefficients in* K *if, and only if, there exist two positive integers* m *and* μ, $m \neq \mu$, *such that* $\alpha_m \alpha_\mu^{-1} \in G$.

Proof. If there exist m and μ, $m \neq \mu$, such that $\alpha_m \alpha_\mu^{-1} \in G$, then $L_i(\alpha_m x)$, $L_i(\alpha_\mu x)$, and 1 are linearly dependent over $K(x)$ by Lemma 3.

Conversely, suppose that

$$\sum_{m=1}^{N} \sum_{i=0}^{k} P_{i,m}(x)\, L_i(\alpha_m x) + Q(x) = 0,$$

where $P_{i,m}, Q \in K[x]$, with $\alpha_m \alpha_\mu^{-1} \notin G$ if $m \neq \mu$.

Then the formal series

$$h(x)=\sum_{m=1}^{N}\sum_{i=0}^{k} P_{i,m}(x)\,L_i(\alpha_m x)=\sum_{n=0}^{+\infty} h_n x^n$$

satisfies $h_n=0$ when n is large enough.

Suppose that at least one of the $P_{i,m}$'s is not zero; put $d=\operatorname{Max}(\deg P_{i,m})$, $d\geqslant 0$. It is readily verified, by using the definition of $L_k(x)$, that

$$h_n=\frac{u_n}{(q^n-1)(q^{n-1}-1)\cdots(q^{n-d}-1)} \qquad (n\geqslant d),$$

where u_n is a linear recurring sequence, of the form

$$u_n=\sum_{n=0}^{d}\sum_{m=1}^{N} Q_{r,m}(n)(q^r\alpha_m)^n,$$

with $Q_{r,m}\in K[x]$.

Choose (i_0, m_0) such that $P_{i_0,m_0}\neq 0$, with i_0 *maximum*. Then Q_{d,m_0} has exact degree i_0, therefore $Q_{d,m_0}\neq 0$. As all the $q^r\alpha_m$'s are different, we have $u_n\neq 0$ when n is large enough, and the Proof of Lemma 5 is complete.

3. EXPLICIT COMPUTATION OF PADÉ–HERMITE APPROXIMANTS BY SIEGEL'S METHOD

Let $\alpha_1, \alpha_2, ..., \alpha_N$ be N elements of K^*, such that $\alpha_m\alpha_\mu^{-1}\notin G$ for $m\neq\mu$ (see Lemma 5).

Let n be an integer.

We want to compute polynomials $P_{\ell,\mu,n}(x)$ and $Q_n(x)$, and a formal series $R_n(x)$, such that $\deg P_{\ell,\mu,n}\leqslant n$, $\deg Q_n\leqslant n$, and

$$\sum_{\mu=1}^{N}\sum_{\ell=0}^{k} P_{\ell,\mu,n}(x)\,L_\ell(\alpha_\mu x)+Q_n(x)=x^{(n+1)\,N(k+1)+n}R_n(x). \qquad (20)$$

Equation (20) can be seen as a homogeneous system of $(n+1)\,N(k+1)$ equations with $(n+1)\,N(k+1)+n+1$ unknowns (the coefficients of Q_n and the $P_{\ell,\mu,n}$'s). Thus (20) admits a nontrivial solution.

First, we show how to compute $R_n(x)$ by using Siegel's method. We q-derive (20) $n+1$ times. Using Leibniz's rule (9), we get:

$$\sum_{\mu=1}^{N}\sum_{\ell=0}^{k}\sum_{j=0}^{n+1}\binom{n+1}{j}_q \alpha_\mu^j(\delta_q^j L_\ell)(\alpha_\mu x)(\delta_q^{n+1-j}P_{\ell,\mu,n})(q^j x)$$
$$= x^{(n+1)N(k+1)-1}S_n(x), \qquad \text{with} \quad S_n(x)\in K[[x]].$$

We observe that $\delta_q^{n+1}P_{\ell,\mu,n}=0$ and that $\delta_q^{n+1-j}P_{\ell,\mu,n}$ is a polynomial of degree $\leqslant j-1$ for $1\leqslant j\leqslant n+1$.

We multiply both sides of the above equality by:

$$\prod_{m=1}^{N}\prod_{v=0}^{n}(1-q^v\alpha_m x)^{k+1}.$$

We obtain:

$$\sum_{\mu=1}^{N}\sum_{\ell=0}^{k}\sum_{j=1}^{n+1}\binom{n+1}{j}_q \alpha_\mu^j(\delta_q^{n+1-j}P_{\ell,\mu,n})(q^j x)(\delta_q^j L_\ell)(\alpha_\mu x)$$
$$\times\prod_{m=1}^{N}\prod_{v=0}^{n}(1-q^v\alpha_m x)^{k+1}=x^{(n+1)N(k+1)-1}T_n(x),$$
$$T_n(x)\in K[[x]]. \tag{21}$$

Now we use Lemma 4: $(\delta_q^j L_\ell)(\alpha_\mu x)\prod_{m=1}^{N}\prod_{v=0}^{n}(1-q^v\alpha_m x)^{k+1}$ is a polynomial of degree less than:

$$N(n+1)(k+1)-j(\ell+1)+\ell(j-1)=N(n+1)(k+1)-j-\ell.$$

Therefore, the left-hand side of (21) is a polynomial of degree less than:

$$\operatorname*{Max}_{\ell}(j-1+N(n+1)(k+1)-j-\ell)=N(n+1)(k+1)-1.$$

But the right-hand side of (21) is a formal series vanishing at zero with order greater than or equal to $N(n+1)(k+1)-1$. Thus we obtain $T_n(x)=c_n\in K$. By Lemma 5, we have $c_n\neq 0$. As we can multiply (20) by any arbitrary constant, we choose $c_n=n_q!$, and we get

$$\left(\prod_{v=0}^{n}\prod_{\mu=1}^{N}(1-q^v\alpha_\mu x)\right)^{k+1}\delta_q^{n+1}(x^{(n+1)N(k+1)+n}R_n(x))$$
$$=n_q!\,x^{(n+1)N(k+1)-1}.$$

Now by Lemma 2, we obtain

$$R_n(x)=\int_0^1 \frac{\prod_{i=1}^n (1-q^i t)\, t^{(n+1)N(k+1)-1}}{\prod_{v=0}^n [\prod_{\mu=1}^N (1-q^v \alpha_\mu tx)]^{k+1}}\, d_q t. \tag{22}$$

In order to get explicit formulas for $P_{\ell,\mu,n}(x)$ and $Q_n(x)$ we start from:

$$W_n(x)=x^{(n+1)N(k+1)+n} R_n(x).$$

Using Cauchy's q-binomial formula (12), we get

$$W_n(x)=x^{(n+1)N(k+1)+n} \sum_{p=0}^{n} (-1)^p \binom{n}{p}_q q^{p(p+1)/2}$$
$$\times \int_0^1 \frac{t^{(n+1)N(k+1)+p-1}}{\prod_{v=0}^n [\prod_{\mu=1}^N (1-q^v \alpha_\mu tx)]^{k+1}}\, d_q t$$
$$W_n(x)=\sum_{p=0}^{n} (-1)^p \binom{n}{p}_q q^{p(p+1)/2} x^{n-p+1}$$
$$\times \int_0^1 \frac{(tx)^{(n+1)N(k+1)+p-1}}{\prod_{v=0}^n [\prod_{\mu=1}^N (1-q^v \alpha_\mu tx)]^{k+1}}\, d_q t$$

We define the polynomials $Q_{n,N,k,p}$ and the numbers $A(v,\mu,\ell,p)$ by the partial fraction expansion:

$$\frac{z^{(n+1)N(k+1)+p-1}}{\prod_{v=0}^n [\prod_{\mu=1}^N (1-zq^v \alpha_\mu)]^{k+1}} = Q_{n,N,k,p}(z)+\sum_{v=0}^{n}\sum_{\mu=1}^{N}\sum_{\ell=0}^{k} \frac{A(v,\mu,\ell,p)}{(1-zq^v\alpha_\mu)^{\ell+1}}. \tag{23}$$

We now obtain:

$$W_n(x)=\sum_{p=0}^{n} (-1)^p \binom{n}{p}_q q^{p(p+1)/2} x^{n-p+1} \times \Bigg(\int_0^1 Q_{n,N,k,p}(tx)\, d_q t$$
$$+\sum_{v=0}^{n}\sum_{\mu=1}^{N}\sum_{\ell=0}^{k} A(v,\mu,\ell,p) \int_0^1 \frac{d_q t}{(1-txq^v\alpha_\mu)^{\ell+1}}\Bigg).$$

The change of variable $u=tx$ in the q-integrals leads to:

$$W_n(x)=\sum_{p=0}^{n} (-1)^p \binom{n}{p}_q q^{p(p+1)/2} x^{n-p}$$
$$\times \Bigg(\int_0^x Q_{n,N,k,p}(u)\, d_q u+\sum_{v=0}^{n}\sum_{\mu=1}^{N}\sum_{\ell=0}^{k} A(v,\mu,\ell,p)\, L_\ell(q^v\alpha_\mu x)\Bigg).$$

We observe that $\int_0^x Q_{n,N,k,p}(u)\,d_q u$ is a polynomial $U_{n,N,k,p}(x)$.
Moreover, by Lemma 3, we have:

$$W_n(x)=\sum_{p=0}^{n}(-1)^p\binom{n}{p}_q q^{p(p+1)/2}x^{n-p}$$
$$\times\left(U_{n,N,k,p}(x)+\sum_{v=0}^{n}\sum_{\mu=1}^{N}\sum_{\ell=0}^{k}A(v,\mu,\ell,p)\right.$$
$$\left.\times\left(L_\ell(\alpha_\mu x)+(q-1)\sum_{\zeta=0}^{v-1}\frac{q^\zeta\alpha_\mu x}{(1-q^\zeta\alpha_\mu x)^{\ell+1}}\right)\right).$$

But the series 1 and $L_\ell(\alpha_\mu x)$ are linearly independent over $K(x)$ by Lemma 5.

If we go back to the expression of $W_n(x)$ in (20), we then see that $P_{\ell,\mu,n}(x)$ is exactly the factor of $L_\ell(\alpha_\mu x)$ in the above equality. Thus we have proved

THEOREM 1. *For every $n\in\mathbb{N}-\{0\}$, we have*

$$\sum_{\mu=1}^{N}\sum_{\ell=0}^{k}P_{\ell,\mu,n}(x)\,L_\ell(\alpha_\mu x)+Q_n(x)=x^{(n+1)N(k+1)+n}R_n(x),$$

where

$$R_n(x)=\int_0^1\frac{\prod_{i=1}^{n}(1-q^it)\,t^{(n+1)N(k+1)-1}}{\prod_{v=0}^{n}[\prod_{\mu=1}^{N}(1-q^v\alpha_\mu tx)]^{k+1}}\,d_qt.$$

$$P_{\ell,\mu,n}(x)=\sum_{p=0}^{n}\sum_{\mu=0}^{n}(-1)^p\binom{n}{p}_q q^{p(p+1)/2}A(v,\mu,\ell,p)\,x^{n-p},$$

the $A(v,\mu,\ell,p)$ being defined in (23).

4. THE CASE $K=\mathbb{C}$

If $K=\mathbb{C}$, it is not difficult to transform the expression of $R_n(x)$ in Theorem 1 into a complex integral.

Suppose $|q|<1$.

If f is any continuous function on $[0,1]$, it is well known [6, Chap. 2; 7, Chap. 1] that:

$$\int_0^1 f(t)\,d_qt=(1-q)\sum_{n=0}^{+\infty}q^nf(q^n). \tag{25}$$

Suppose now that f is analytic in $D=\{|z|<1/|q|\}$, and let $\mathscr{C}$ be any positive contour included in D and enclosing all the numbers q^n, $n\in\mathbb{N}$. If we use the residue theorem, we immediately obtain:

$$\int_0^1 f(t)\,d_q t=\frac{1-q}{2i\pi}\int_{\mathscr{C}} f(z)\sum_{n=0}^{+\infty}\frac{q^n}{z-q^n}\,dz. \tag{26}$$

Thus, if $|x|<|q^{-n}|\ \mathrm{Min}|\alpha_\mu^{-1}|$ and $|q|<1$:

$$R_n(x)=\frac{1-q}{2i\pi}\int_{\mathscr{C}}\frac{\prod_{\nu=1}^{n}(1-q^i z)\,z^{(n+1)N(k+1)-1}}{\prod_{\nu=0}^{n}\left[\prod_{\mu=1}^{N}(1-q^\nu\alpha_\mu zx)\right]^{k+1}}\sum_{n=0}^{+\infty}\frac{q^n}{z-q^n}\,dz.$$

We can obtain a similar expression if $|q|>1$ if we use (see [5, Theorem 7]):

$$\int_0^1 f(t)\,d_q t=(q-1)\sum_{n=0}^{+\infty}\frac{1}{q^{n+1}}f\left(\frac{1}{q^{n+1}}\right). \tag{28}$$

If $\mathscr{C}$ is any positive contour enclosing the q^{-n}, $n\in\mathbb{N}$, and if $|x|<|q^{-n}|\ \mathrm{Min}|\alpha_\mu^{-1}|$ and $|q|>1$, we get:

$$R_n(x)=\frac{q-1}{2i\pi}\int_{\mathscr{C}}\frac{\prod_{i=1}^{n}(1-q^i z)\,z^{(n+1)N(k+1)-1}}{\prod_{\nu=0}^{n}\left[\prod_{\mu=1}^{N}(1-q^\nu\alpha_\mu zx)\right]^{k+1}}\sum_{n=0}^{+\infty}\frac{1}{q^{n+1}z-1}\,dz. \tag{29}$$

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REFERENCES

1. P. Borwein, Padé approximants for the q-elementary functions, *Constr. Approx.* **4** (1988), 391–402.
2. P. Borwein, On the irrationality of $\sum 1/(q^n+r)$, *J. Number Theory* **37** (1991), 253–259.
3. P. Borwein, On the irrationality of certain series, *Math. Proc. Cambridge Philos. Soc.* **112** (1992), 141–146.
4. D. Duverney, U-dérivation, *Ann. Fac. Sci. Toulouse* **2**, No. 3 (1993), 323–335.
5. D. Duverney, Approximants de Padé et U-dérivation, *Bull. Soc. Math. France* **122** (1994), 553–570.
6. H. Exton, "q-Hypergeometric Functions and Applications," Chichester, 1983.
7. G. Gasper and M. Rahman, "Basic Hypergeometric Series," Cambridge Univ. Press, Cambridge, 1990.
8. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," 5th ed., Oxford Science Publications, Oxford, 1979.
9. M. Huttner, Approximations rationnelles du dilogarihme, preprint.

10. H. Jager, A multidimensional generalization of the Padé table, *Nederl. Akad. Wetensch. Proc. Ser. A* **67** (1964), 227–249.
11. K. Mahler, Zur Approximation der Exponentialfunktion und des Logarithmus I und II, *J. Reine Angew. Math.* **166** (1932), 118–150.
12. C. L. Siegel, Transcendental numbers, Princeton, NJ, 1949.
13. A. B. Shidlovskii, "Transcendental Numbers," de Gruyter, Berlin, 1989.
14. R. Wallisser, Rationale Approximation des q-Analogons der Exponentialfunktion und Irrationalitäsaussagen für diese Funktion, *Arch. Math.* **44** (1985), 59–64.